

Strongly transitive multiple trees

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Abstract

We give an amalgamation construction of free multiple trees with a strongly transitive automorphism group. The construction shows that any partial codistance function on a tuple of finite trees can be extended to yield multiple trees.

1 Introduction

Multiple trees are a generalization of twin trees and buildings. While twin buildings arise from certain Kac-Moody groups, multiple trees were introduced by Tits and Ronan in order to deal with several valuations at the same time.

Since multiple trees are more rigid than twin trees, it is natural to ask whether as in the case of spherical buildings of rank at least three a complete classification might be possible at least under the assumption that the automorphism group be strongly transitive. The construction given here shows that this is not the case. Furthermore, any codistance function on a tuple of finite trees can be extended to yield multiple trees of infinite valency with a BN-pair.

Free constructions of twin trees were given in [5]. However that construction started from generalized polygons and did not yield multiple trees: for more than two trees, the definition of multiple trees entails a certain regularity between sets of pairwise opposite vertices which cannot hold in generalized n -gons for $n > 6$. For this reason, the construction given here is completely different from the one for twin trees given in [5]. Mühlherr and Struyve informed me that they have a construction for free multiple trees similar to

the free construction of polygons. However, their construction does not yield strong transitivity.

2 Construction

Recall that given an infinite tree T without end-vertices, a codistance on T is a mapping d^* from T to the set \mathbb{N} of nonnegative integers, such that, if $d^*(v) = n$ and $v' \sim v$ in T , then $d^*(v') \in \{n-1, n+1\}$. Moreover, if $n > 0$ then $d^*(v') = n+1$ for a unique $v' \sim v$ where we write $x \sim y$ if x is a neighbour of y . Given a (possibly infinite) family $\{T_i\}_{i \in I}$ of trees, a multiple tree over $\{T_i\}_{i \in I}$ is defined by a codistance function $d: \prod_{i \in I} T_i \rightarrow \mathbb{N}$ such that, for any choice of $k \in I$ and any $\bar{a} = (a_i)_{i \in I} \in \prod_{i \in I} T_i$, the function $d_{\bar{a};k}^*$ induced by d on the graph $\{(x_h)_{h \in I} \mid x_h = a_h \text{ for } h \neq k\} \cong T_k$ is a codistance on T_k . It will also be convenient to denote by $d_{\bar{a};i,j}^*$ the codistance induced on the graph $\{(x_h)_{h \in I} \mid x_h = a_h \text{ for } h \neq i, j\} \cong T_i \times T_j$

Remark 2.1. Suppose that $d^*(x_1, \dots, x_n) = k$ and that $y_i \sim x_i, y_j \sim x_j$ are such that $d_{\bar{x};i,j}^*(y_i, x_j) = k+1 = d_{\bar{x};i,j}^*(x_i, y_j)$. Then it follows easily that we have $d_{\bar{x};i,j}^*(y_i, y_j) = k+2$.

Definition 2.2. Let \mathcal{K}_n be the class of n -tuples of nonempty finite trees

$$((A_1, \dots, A_n), d)$$

with a codistance function $d: \prod_{i=1, \dots, n} A_i \rightarrow \mathbb{N}$ such that the following holds

1. there are $x_i \in A_i, i = 1, \dots, n$ with $d^*(x_1, \dots, x_n) = 0$;
2. if $d^*(x_1, \dots, x_n) = k$, then for each $i = 1, \dots, n$ and any $y \sim x_i$ we have $d_{\bar{x};i}^*(y) \in \{k+1, k-1\}$.
3. if $d^*(x_1, \dots, x_n) = k > 0$, then for each $i = 1, \dots, n$ there is at most one $y \sim x_i$ with $d_{\bar{x};i}^*(y) = k+1$.
4. if $d^*(x_1, \dots, x_n) = k$ and $y_i \sim x_i, y_j \sim x_j$ are such that $d_{\bar{x};i,j}^*(y_i, x_j) = k+1 = d_{\bar{x};i,j}^*(x_i, y_j)$, then $d_{\bar{x};i,j}^*(y_i, y_j) = k+2$.

Definition 2.3. Let $\mathcal{A} = (\bar{A}, d^*)$ be in \mathcal{K}_n . We call $(x_1, \dots, x_n), (y_1, \dots, y_n)$ in \mathcal{A} geodesic if

$$d^*(y_1, \dots, y_n) = d^*(x_1, \dots, x_n) - D$$

where $D = \sum_{j \leq n} \text{dist}(x_j, y_j)$ and dist denotes the graph theoretic distance.

Lemma 2.4. Let $\mathcal{A} = (\bar{A}, d)$ be in \mathcal{K}_n . If (x_1, \dots, x_n) and (y_1, \dots, y_n) are geodesic, then we have

$$d^*(y_1, \dots, y_n) \leq d^*(a_1, \dots, a_n) \leq d^*(x_1, \dots, x_n)$$

if and only if all $a_j \in A_j, j = 1, \dots, n$ are on the geodesic from y_j to x_j

Proof. This follows immediately from the definition. \square

Let $\mathcal{A} = (\bar{A}, d)$ be in \mathcal{K}_n . We say that d^* is *locally maximal* in (x_1, \dots, x_n) if there is no $y \sim x_i$ for any $i = 1, \dots, n$ such that

$$d_{\bar{x};i}^*(y) > d_{\bar{x};i}^*(x_i).$$

Definition 2.5. Let $\mathcal{A} = (\bar{A}, d^*)$ be in \mathcal{K}_n . The following 1-point extensions are called *elementary good extensions* of \mathcal{A} :

1. add a vertex y to A_k for some $1 \leq k \leq n$ with $y \sim x_k \in A_k$ and for any $x_i \in A_i, i \neq k$ put

$$d_{\bar{x};k}^*(y) = |d_{\bar{x};k}^*(x_k) - 1|.$$

2. if $d^*(x_1, \dots, x_n)$ is locally maximal in A add a vertex y to A_k with $y \sim x_k$ and extend d^* to the extension as follows:

If $(y_1, \dots, y_{k-1}, x_k, y_{k+1}, \dots, y_n)$ is geodesic with (x_1, \dots, x_n) , then

$$d_{\bar{y};k}^*(y) = d_{\bar{x};k}^*(x_k) + 1.$$

and otherwise

$$d_{\bar{y};k}^*(y) = |d_{\bar{x};k}^*(x_k) - 1|.$$

For $A, B \in \mathcal{K}_n$ we say that A is good in B if B arises from A by a finite sequence of elementary good extensions.

Lemma 2.6. Suppose that $(A, d^*) \in \mathcal{K}_n$ and $(A \cup \{y\}, d^*)$ is an elementary good extension. Then $(A \cup \{y\}, d^*) \in \mathcal{K}_n$.

Proof. If the extension is of type 1. all conditions continue to hold automatically. We have to show that the conditions hold for extensions of type 2. Conditions 1. and 2. are still clear.

For 3. we have to show that for any $(a_1, \dots, a_n) \in A \cup \{y\}$ and $1 \leq i \leq n$ there is at most one $x \sim a_i$ with $d_{\bar{a};i}^*(x) > d_{\bar{a};i}^*(a_i)$. Suppose that $d^*(x_1, \dots, x_n)$

is locally maximal in A and a vertex y was attached to A_k with $y \sim x_k$. If $a_k \neq x_k, y$, then all vertices are inside A and since d^* was not changed on A , the claim remains true.

So suppose $a_k = x_k$. If $\bar{a} = \bar{x}$, then the claim follows from the local maximality. Hence we may assume that for some $1 \leq j \leq n$ we have $a_j \neq x_j$ (and clearly $j \neq k$). Since there is a unique new vertex y , we only have to consider the case $i = k$ and $d_{\bar{a};k}^*(y) > d_{\bar{a};k}^*(a_i)$. Then (a_1, \dots, a_n) is geodesic with (x_1, \dots, x_n) . Thus for $b \sim a_j$ in the interval $[a_j, x_j]$ we have

$$d_{\bar{a};i}^*(b) = d^*(a_i) + 1.$$

Suppose that for some further $x \sim x_k = a_k, x \neq y$ we have

$$d_{\bar{a};k}^*(x) = d_{\bar{a};k}^*(a_k) + 1.$$

Since $A \in \mathcal{K}_n$ this implies

$$d_{\bar{a};k,j}^*(x, b) = d_{\bar{a};k,j}^*(x_k, a_j) + 2$$

and hence

$$d_{\bar{a};k,j}^*(x_k, b) \geq d_{\bar{a};k,j}^*(x_k, a_j) + 1$$

contradicting the locally maximal choice of $d^*(x_1, \dots, x_n)$ (remember that $a_k = x_k$).

Finally consider the case $a_k = y$. For $i = k$ there is nothing to show since y has a unique neighbour, so suppose $i \neq k$. If

$$d_{\bar{a};k}^*(y) > d_{\bar{a};k}^*(x_k)$$

then $(a_1, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) .

If $x \sim a_i$ is such that

$$d_{\bar{a};i}^*(x) = d_{\bar{a};i,k}^*(x, y) > d_{\bar{a};i}^*(a_i) = d_{\bar{a};k}^*(y) > d_{\bar{a};k}^*(x_k)$$

then

$$d_{\bar{a};i,k}^*(x, y) > d_{\bar{a};i,k}^*(a_i, y) = d_{\bar{a};k}^*(y) > d_{\bar{a};i,k}^*(a_i, x_k)$$

and therefore

$$d_{\bar{a};i,k}^*(x, y) > d_{\bar{a};i,k}^*(x, x_k).$$

Hence by Lemma 2.4 x is the unique neighbour of a_i closer to x_i .

Next suppose

$$d_{\bar{a};k}^*(y) < d_{\bar{a};k}^*(x_k)$$

and that there are $b_1, b_2 \sim a_i$ with

$$d_{\bar{a};i}^*(b_1) = d_{\bar{a};i}^*(b_2) > d_{\bar{a};i}^*(a_i).$$

If

$$d_{\bar{a};i,k}^*(b_1, x_k) = d_{\bar{a};i,k}^*(b_2, x_k) < d_{\bar{a};i,k}^*(a_i, x_k) = d_{\bar{a};k}^*(x_k)$$

then

$$d_{\bar{a};i}^*(y) = d_{\bar{a};i,k}^*(b_1, x_k) = d_{\bar{a};i,k}^*(b_2, x_k) < d_{\bar{a};i,k}^*(b_1, y) = d_{\bar{a};i,k}^*(b_2, y).$$

Then $(a_1 \dots a_{i-1}, b_s, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) for $s = 1, 2$. But this clearly implies that also

$$(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$$

is geodesic with (x_1, \dots, x_n) , contradicting

$$d_{\bar{a};k}^*(y) < d_{\bar{a};k}^*(x_k)$$

As A is in \mathcal{K}_n we may therefore assume

$$d_{\bar{a};i,k}^*(b_1, x_k) < d_{\bar{a};i,k}^*(a_i, x_k) < d_{\bar{a};i,k}^*(b_2, x_k).$$

Then

$$d_{\bar{a};i,k}^*(b_1, x_k) = d_{\bar{a};i,k}^*(a_i, y) < d_{\bar{a};i,k}^*(b_1, y) = d_{\bar{a};i,k}^*(b_2, y) < d_{\bar{a};i,k}^*(b_2, x_k).$$

This implies that $(a_1 \dots a_{i-1}, b_2, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) from which we again conclude that also

$$(a_1 \dots a_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$$

is geodesic with (x_1, \dots, x_n) , a contradiction.

It remains to prove that Condition 4. holds: if $d^*(a_1, \dots, a_n) = m$ and $y_i \sim a_i, y_j \sim a_j$ are such that $d_{\bar{a};i,j}^*(y_i, a_j) = m + 1 = d_{\bar{a};i,j}^*(a_i, y_j)$, then $d_{\bar{a};i,j}^*(y_i, y_j) = m + 2$.

Since $A \in \mathcal{K}_n$ we only have to check the situation when $a_k = y$ or when $a_k = x_k$ and $i = k$.

First assume $a_k = x_k$ and $i = k$. Assume $y \sim x_k, y_j \sim a_j$ are such that $d_{\bar{a};k,j}^*(y, a_j) = m + 1 = d_{\bar{a};k,j}^*(x_k, y_j)$. then (a_1, \dots, a_n) is geodesic with (x_1, \dots, x_n) . Lemma 2.4 implies that $y_j \sim a_j$ lies between a_j and x_j and hence $(a_1, \dots, a_{j-1}, y_j, a_{j+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) . Hence $d_{\bar{a};k,j}^*(y, y_j) = m + 2$.

Next we assume $a_k = y$ and $i = k$. Assume $y_k = x_k \sim y, y_j \sim a_j$ are such that $d_{\bar{a};k,j}^*(x_k, a_j) = m + 1 = d_{\bar{a};k,j}^*(y, y_j)$. If $d_{\bar{a};k,j}^*(x_k, y_j) < d_{\bar{a};k,j}^*(y, y_j)$, then $(a_1, \dots, a_{k-1}, x_k, a_{k+1}, \dots, y_j, a_{j+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) . Now $d_{\bar{a};k,j}^*(x_k, a_j) = m + 1$ implies that also $(a_1, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_j, a_{j+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) , and hence $d_{\bar{a};k,j}^*(x_k, a_j) < d_{\bar{a};k,j}^*(y, a_j)$, a contradiction.

Now assume $a_k = y$ and $i, j \neq k$. Let $y_i \sim a_i, y_j \sim a_j$ be such that $d_{\bar{a};i,j}^*(y_i, a_j) = m + 1 = d_{\bar{a};i,j}^*(a_i, y_j)$. If $d_{\bar{a};i,j}^*(y_i, y_j) < d_{\bar{a};i,j}^*(a_i, y_j)$, then $(a_1, \dots, y_j, a_{j+1}, \dots, a_n)$ and hence (a_1, \dots, a_n) are geodesic with (x_1, \dots, x_n) (while $(a_1, \dots, y_i, a_{i+1}, \dots, a_n)$ is not geodesic with (x_1, \dots, x_n) .) Thus for the unique $b_i \sim a_i$ between a_i and x_i we have $d_{\bar{a};i,j}^*(b_i, y_j) = m + 2$. But since $A \cup \{y\}$ satisfies Condition 2. we must have $y_i = b_i$, a contradiction. \square

Lemma 2.7. *Suppose that $(A, d^*), (A \cup \{y\}, d^*) \in \mathcal{K}_n$. Then $A \cup \{y\}$ is an elementary good extension of A .*

Proof. Let $y \sim x_k$ for some (unique) $x_k \in A_k$. If $d_{\bar{x};k}^*(y) < d_{\bar{x};k}^*(x_k)$ for all $(x_1, \dots, x_n) \in A$, then $A \cup \{y\}$ is an extension of type 1. So suppose for some $(x_1, \dots, x_n) \in A$ we have $d_{\bar{x};k}^*(y) > d_{\bar{x};k}^*(x_k)$, which we may assume to be locally maximal. Let $(z_1, \dots, z_n) \in A$ with $z_k = x_k$ and $d_{\bar{z};k}^*(y) > d_{\bar{z};k}^*(x_k)$. We have to show that \bar{z} and \bar{x} are geodesic. Otherwise for some j on the path $\gamma = (x_j = y_0, \dots, y_m = z_j)$ there is some y_s such that

$$d_{\bar{x};j}^*(y_s) < d_{\bar{x};j}^*(y_{s-1}, d_{\bar{x};j}^*(y_{s+1}$$

contradicting Axiom 3. \square

Corollary 2.8. *Suppose that $(A, d^*), (B, d^*) \in \mathcal{K}_n$ with $A \subseteq B$. Then A is good in B .* \square

Lemma 2.9. *The class \mathcal{K}_n has the amalgamation property.*

Proof. Clearly it suffices inductively to prove this for $A \subseteq B, C$ where B, C are elementary good extensions of A , so $B = A \cup \{b\}, C = A \cup \{c\}, b \in B_i, c \in C_j$ with unique neighbours b', c' .

If $i = j$ and at least one of the extensions is of type 1., then $A \cup \{b, c\}$ with the codistance induced by B, C is in \mathcal{K}_n . So suppose both extensions are of type 2. In this case $A \cup \{b, c\}$ with the codistance induced by B, C is in \mathcal{K}_n unless $b' = c'$ and b' and c' increase the distance to the same locally maximal tuple. In this case we identify b and c and choose B as the amalgam.

Now suppose $i < j$. If B is an elementary good extension of A of the first kind, define d^* on $A \cup \{b, c\}$ by

$$d_{\bar{x};i,j}^*(b, c) = d_{\bar{x};i,j}^*(b', c) - 1.$$

Then $A \cup \{b, c\}$ is an elementary good extension of C of the first kind and is therefore in \mathcal{K}_n by Lemma 2.6.

Next suppose that B and C are elementary good extensions of A of the second kind and that b, c were attached to different locally maximal tuples. Let $(y_1, \dots, y_{i-1}, b', y_{i+1}, \dots, y_n)$ be the locally maximal tuples to which b was attached. If $(x_1, \dots, x_{i-1}, b', x_{i+1}, \dots, x_{j-1}, c, x_{j+1}, \dots, x_n)$ is geodesic with $(y_1, \dots, y_{i-1}, b', y_{i+1}, \dots, y_n)$ define d^* on $A \cup \{b, c\}$ by

$$d_{\bar{x};i,j}^*(b, c) = d_{\bar{x};i,j}^*(b', c) + 1.$$

and otherwise

$$d_{\bar{y};i,j}^*(b, c) = |d_{\bar{y};i,j}^*(b', c) - 1|.$$

Then $A \cup \{b, c\}$ is an elementary good extension of C of the second kind and is therefore in \mathcal{K}_n by Lemma 2.6.

Finally suppose that b, c are attached to the same locally maximal tuple $(x_1, \dots, x_{i-1}, b', x_{i+1}, \dots, x_{j-1}, c', x_{j+1}, \dots, x_n)$.

If $(y_1, \dots, y_{i-1}, b', y_{i+1}, \dots, y_{j-1}, c', y_{j+1}, \dots, y_n)$ is geodesic with

$$(x_1, \dots, x_{i-1}, b', x_{i+1}, \dots, x_{j-1}, c', x_{j+1}, \dots, x_n),$$

then

$$d_{\bar{y};i,j}^*(b, c) = d_{\bar{x};i,j}^*(b, c') + 1.$$

and otherwise

$$d_{\bar{y};i,j}^*(b, c) = |d_{\bar{y};i,j}^*(b, c') - 1|.$$

Then $A \cup \{b, c\}$ is an elementary good extension of C of the second kind and is therefore in \mathcal{K}_n by Lemma 2.6.

□

Using a bit of model theory, we may extend the language of graphs with n -ary predicates $d_k^*, k \in \mathbb{N}$ denoting the codistance and by binary function symbols $f_i(x, y), i \in \mathbb{N}$ with $f_i(x, y) = z$ if z is the i^{th} element on the path from x to y if such a path of length at least i exists and $z = x$ otherwise. In this expanded language, the substructure generated by a subset A will include all finite geodesic paths between elements from the same coordinate tree A_k . Hence in this language, the class \mathcal{K}_n is closed under finitely generated substructures. Since by Axiom 1. any $A \in \mathcal{K}_n$ contains the structure consisting only of an n -tuple (x_1, \dots, x_n) with $d^*(\bar{x}) = 0$, the class \mathcal{K}_n has a Fraïssé limit M_n (see e.g. [6], Ch. 4.4), i.e. a countable structure whose finite substructures are exactly the structures isomorphic to elements of \mathcal{K}_n and whose automorphism group acts transitively on isomorphism classes of finite substructures.

Suppose A is an n -fold tree and $(x_1, \dots, x_n) \in A$ is such that $d^*(\bar{x}) = 0$. and $y, z \sim x_i$ for some $1 \leq i \leq n$. Then the set

$$\mathbb{T}_{\bar{x}, y} = \bigcup_{1 \leq j \leq n} \{z \in A_j : d_{\bar{x}; i, j}^*(y, z) > d_{\bar{x}; i, j}^*(x_i, z)\}$$

is called a *half apartment*, and the set

$$\mathbb{T}_{\bar{x}; y, z} = \mathbb{T}_{\bar{x}; y} \cup \mathbb{T}_{\bar{x}; z}$$

is called an *apartment*.

The automorphism group of a multiple tree $A = \{A_i\}_{i \leq n}$ is said to act *strongly transitively* if it acts transitively on the set of *marked apartments*, i.e. transitively on the set $\{(x_1, \dots, x_n, y) : x_i \in A_i, y \sim x_1\}$. This is equivalent to the automorphism group having a BN-pair (see [1] Ch. 6).

Hence we obtain our main theorem:

Theorem 2.1. *The Fraïssé limit M_n is a multiple tree whose automorphism group acts strongly transitively on the set of apartments.*

Proof. It is clear from the construction that if $d_{\bar{x}; i}^*(x) = k$ then there exists a unique $y \sim x_i$ such that $d_{\bar{x}; i}^*(y) = k + 1$.

The strong transitivity of $\text{Aut}(M_n)$ is immediate by the properties of the Fraïssé limit: the structure $\{(x_1, \dots, x_n)\} \cup \{y\}$ with $y \sim x_1$ is in \mathcal{K}_n and $\text{Aut}(M_n)$ acts transitively on the set of substructures of M_n isomorphic to it. \square

For a half apartment $\mathbb{T}_{\bar{x};y}$ the corresponding *root group* $U_{\mathbb{T},\bar{x};y}$ in $\text{Aut}(A)$ is the subgroup of $\text{Aut}(A)$ fixing the set $\mathbb{T}_{\bar{x};y} \cup \{v: v \sim z \text{ for some } z \in \mathbb{T}_{\bar{x};y}\}$ pointwise.

We say that a multiple tree satisfies the *Moufang condition* if for each root the corresponding root group acts transitively on the set of apartments containing the given root.

Remark 2.10. *It is easy to see that if a root group acts transitively on the set of apartments containing the given half-apartment, then it acts regularly. In other words, the stabilizer of an apartment inside a root group is trivial (see [4], Sec. 4).*

Proposition 2.11. *The automorphism group of the multiple tree M_n does not satisfy the Moufang condition.*

Proof. We claim that in fact all root groups are trivial. Note that by the properties of the Fraïssé limit, if (x_1, \dots, x_n) in M_n is such that $d^*(x_1, \dots, x_n) = 0$, the stabilizer H of x_1, \dots, x_n acts highly transitively (i.e. m -transitively for any m) on the set of vertices $z \sim x_i$ since for any $m \in \mathbb{N}$ the structure $\{x_1, \dots, x_n, z_1, \dots, z_m\}$ with $z_j \sim x_i, j = 1, \dots, m$ is in \mathcal{K}_n and d^* is uniquely determined. In particular, the stabilizer H_y of y in H acts highly transitively on the set of $z \sim x_i, z \neq y$ and normalizes $U_{\mathbb{T},\bar{x};y}$. If $U_{\mathbb{T},\bar{x};y}$ was nontrivial, this would contradict Remark 2.10, whence the claim. □

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I thank Max Horn for pointing out an error in an earlier version of the paper.

2 Construction

Recall that given an infinite tree T without end-vertices, a codistance on T is a mapping d^* from T to the set \mathbb{N} of nonnegative integers, such that, if $d^*(v) = n$ and $v' \sim v$ in T , then $d^*(v') \in \{n - 1, n + 1\}$. Moreover, if $n > 0$ then $d^*(v') = n + 1$ for a unique $v' \sim v$ where we write $x \sim y$ if x is a neighbour of y . Given a (possibly infinite) family $\{T_i\}_{i \in I}$ of trees, a multiple tree over $\{T_i\}_{i \in I}$ is defined by a codistance function $d: \prod_{i \in I} T_i \rightarrow \mathbb{N}$ such that, for any choice of $k \in I$ and any $\bar{a} = (a_i)_{i \in I} \in \prod_{i \in I} T_i$, the function $d_{\bar{a};k}^*$ induced by d on the graph $\{(x_h)_{h \in I} \mid x_h = a_h \text{ for } h \neq k\} \cong T_k$ is a codistance on T_k . It will also be convenient to denote by $d_{\bar{a};i,j}^*$ the codistance induced on the graph $\{(x_h)_{h \in I} \mid x_h = a_h \text{ for } h \neq i, j\} \cong T_i \times T_j$

Remark 2.1. Suppose that $d^*(x_1, \dots, x_n) = k > 0$ and that $y_i \sim x_i, y_j \sim x_j$ are such that $d_{\bar{x};i,j}^*(y_i, x_j) = k + 1 = d_{\bar{x};i,j}^*(x_i, y_j)$. Then it follows easily that we have $d_{\bar{x};i,j}^*(y_i, y_j) = k + 2$.

Definition 2.2. Let \mathcal{K}_n be the class of n -tuples of nonempty finite trees

$$((A_1, \dots, A_n), d)$$

with a codistance function $d: \prod_{i=1, \dots, n} A_i \rightarrow \mathbb{N}$ such that the following holds

1. there are $x_i \in A_i, i = 1, \dots, n$ with $d^*(x_1, \dots, x_n) = 0$;
2. if $d^*(x_1, \dots, x_n) = k$, then for each $i = 1, \dots, n$ and any $y \sim x_i$ we have $d_{\bar{x};i}^*(y) \in \{k + 1, k - 1\}$.
3. if $d^*(x_1, \dots, x_n) = k > 0$, then for each $i = 1, \dots, n$ there is at most one $y \sim x_i$ with $d_{\bar{x};i}^*(y) = k + 1$.
4. if $d^*(x_1, \dots, x_n) = k > 0$ and $y_i \sim x_i, y_j \sim x_j$ are such that $d_{\bar{x};i,j}^*(y_i, x_j) = k + 1 = d_{\bar{x};i,j}^*(x_i, y_j)$, then $d_{\bar{x};i,j}^*(y_i, y_j) = k + 2$.

Definition 2.3. Let $\mathcal{A} = (\bar{A}, d^*)$ be in \mathcal{K}_n . We call $(x_1, \dots, x_n), (y_1, \dots, y_n)$ in A geodesic if

$$d^*(y_1, \dots, y_n) = d^*(x_1, \dots, x_n) - D$$

where $D = \sum_{j \leq n} \text{dist}(x_j, y_j)$ and dist denotes the graph theoretic distance.

Lemma 2.4. Let $\mathcal{A} = (\bar{A}, d)$ be in \mathcal{K}_n . If (x_1, \dots, x_n) and (y_1, \dots, y_n) are geodesic, then we have

$$d^*(y_1, \dots, y_n) \leq d^*(a_1, \dots, a_n) \leq d^*(x_1, \dots, x_n)$$

if all $a_j \in A_j, j = 1, \dots, n$ are on the geodesic from y_j to x_j . If $d^*(y_1, \dots, y_n) > 0$ and $a_j \sim y_j$, then

$$d_{\bar{y},j}^*(y_j) \leq d_{\bar{y},j}^*(a_j) \leq d_{\bar{y},j}^*(x_j)$$

if and only if a_j is on the geodesic from y_j to x_j .

Proof. The first part follows immediately from the definition, the second part follows from Axiom 3. \square

Let $\mathcal{A} = (\bar{A}, d)$ be in \mathcal{K}_n . We say that d^* is *locally maximal* in (x_1, \dots, x_n) if there is no $y \sim x_i$ for any $i = 1, \dots, n$ such that

$$d_{\bar{x};i}^*(y) > d_{\bar{x};i}^*(x_i).$$

Definition 2.5. Let $\mathcal{A} = (\bar{A}, d^*)$ be in \mathcal{K}_n . The following 1-point extensions are called *elementary good extensions* of \mathcal{A} :

1. add a vertex y to A_k for some $1 \leq k \leq n$ with $y \sim x_k \in A_k$ and for any $x_i \in A_i, i \neq k$ put

$$d_{\bar{x};k}^*(y) = |d_{\bar{x};k}^*(x_k) - 1|.$$

2. if $d^*(x_1, \dots, x_n)$ is locally maximal in A add a vertex y to A_k with $y \sim x_k$ and extend d^* to the extension as follows:

If $(y_1, \dots, y_{k-1}, x_k, y_{k+1}, \dots, y_n)$ is geodesic with (x_1, \dots, x_n) , then

$$d_{\bar{y};k}^*(y) = d_{\bar{x};k}^*(x_k) + 1.$$

and otherwise

$$d_{\bar{y};k}^*(y) = |d_{\bar{x};k}^*(x_k) - 1|.$$

For $A, B \in \mathcal{K}_n$ we say that A is good in B if B arises from A by a finite sequence of elementary good extensions.

Lemma 2.6. *Suppose that $(A, d^*) \in \mathcal{K}_n$ and $(A \cup \{y\}, d^*)$ is an elementary good extension. Then $(A \cup \{y\}, d^*) \in \mathcal{K}_n$.*

Proof. If the extension is of type 1. all conditions continue to hold automatically. We have to show that the conditions hold for extensions of type 2. Conditions 1. and 2. are still clear.

For 3. we have to show that for any $(a_1, \dots, a_n) \in A \cup \{y\}$ with $d^*(a_1, \dots, a_n) > 0$ and $1 \leq i \leq n$ there is at most one $x \sim a_i$ with $d_{\bar{a};i}^*(x) > d_{\bar{a};i}^*(a_i)$. Suppose that $d^*(x_1, \dots, x_n)$ is locally maximal in A and a vertex y was attached to A_k with $y \sim x_k$. If $a_k \neq x_k, y$, then all vertices are inside A and since d^* was not changed on A , the claim remains true.

So suppose $a_k = x_k$. If $\bar{a} = \bar{x}$, then the claim follows from the local maximality. Hence we may assume that for some $1 \leq j \leq n$ we have $a_j \neq x_j$ (and clearly $j \neq k$). Since there is a unique new vertex y , we only have to consider the case $i = k$ and $d_{\bar{a};k}^*(y) > d_{\bar{a};k}^*(a_i)$. Then (a_1, \dots, a_n) is geodesic with (x_1, \dots, x_n) . Thus for $b \sim a_j$ in the interval $[a_j, x_j]$ we have

$$d_{\bar{a};i}^*(b) = d^*(a_i) + 1.$$

Suppose that for some further $x \sim x_k = a_k, x \neq y$ we have

$$d_{\bar{a};k}^*(x) = d_{\bar{a};k}^*(a_k) + 1.$$

Since $A \in \mathcal{K}_n$ this implies

$$d_{\bar{a};k,j}^*(x, b) = d_{\bar{a};k,j}^*(x_k, a_j) + 2$$

and hence

$$d_{\bar{a};k,j}^*(x_k, b) \geq d_{\bar{a};k,j}^*(x_k, a_j) + 1$$

contradicting the locally maximal choice of $d^*(x_1, \dots, x_n)$ (remember that $a_k = x_k$).

Finally consider the case $a_k = y$. For $i = k$ there is nothing to show since y has a unique neighbour, so suppose $i \neq k$. If

$$d_{\bar{a};k}^*(y) > d_{\bar{a};k}^*(x_k)$$

then $(a_1, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) .

If $x \sim a_i$ is such that

$$d_{\bar{a};i}^*(x) = d_{\bar{a};i,k}^*(x, y) > d_{\bar{a};i}^*(a_i) = d_{\bar{a};k}^*(y) > d_{\bar{a};k}^*(x_k)$$

then

$$d_{\bar{a};i,k}^*(x, y) > d_{\bar{a};i,k}^*(a_i, y) = d_{\bar{a};k}^*(y) > d_{\bar{a};i,k}^*(a_i, x_k)$$

and therefore

$$d_{\bar{a};i,k}^*(x, y) > d_{\bar{a};i,k}^*(x, x_k).$$

Hence by Lemma 2.4 x is the unique neighbour of a_i closer to x_i .

Next suppose

$$d_{\bar{a};k}^*(y) < d_{\bar{a};k}^*(x_k)$$

and that there are $b_1, b_2 \sim a_i$ with

$$d_{\bar{a};i}^*(b_1) = d_{\bar{a};i}^*(b_2) > d_{\bar{a};i}^*(a_i).$$

If

$$d_{\bar{a};i,k}^*(b_1, x_k) = d_{\bar{a};i,k}^*(b_2, x_k) < d_{\bar{a};i,k}^*(a_i, x_k) = d_{\bar{a};k}^*(x_k)$$

then

$$d_{\bar{a};i}^*(y) = d_{\bar{a};i,k}^*(b_1, x_k) = d_{\bar{a};i,k}^*(b_2, x_k) < d_{\bar{a};i,k}^*(b_1, y) = d_{\bar{a};i,k}^*(b_2, y).$$

Then $(a_1, \dots, a_{i-1}, b_s, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) for $s = 1, 2$. But this clearly implies that also

$$(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$$

is geodesic with (x_1, \dots, x_n) , contradicting

$$d_{\bar{a};k}^*(y) < d_{\bar{a};k}^*(x_k)$$

As A is in \mathcal{K}_n we may therefore assume

$$d_{\bar{a};i,k}^*(b_1, x_k) < d_{\bar{a};i,k}^*(a_i, x_k) < d_{\bar{a};i,k}^*(b_2, x_k).$$

Then

$$d_{\bar{a};i,k}^*(b_1, x_k) = d_{\bar{a};i,k}^*(a_i, y) < d_{\bar{a};i,k}^*(b_1, y) = d_{\bar{a};i,k}^*(b_2, y) < d_{\bar{a};i,k}^*(b_2, x_k).$$

This implies that $(a_1 \dots a_{i-1}, b_2, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) from which we again conclude that also

$$(a_1 \dots a_{i-1}, a_i, a_{i+1}, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_n)$$

is geodesic with (x_1, \dots, x_n) , a contradiction.

It remains to prove that Condition 4. holds: if $d^*(a_1, \dots, a_n) = m > 0$ and $y_i \sim a_i, y_j \sim a_j$ are such that $d_{\bar{a};i,j}^*(y_i, a_j) = m + 1 = d_{\bar{a};i,j}^*(a_i, y_j)$, then $d_{\bar{a};i,j}^*(y_i, y_j) = m + 2$.

Since $A \in \mathcal{K}_n$ we only have to check the situation when $a_k = y$ or when $a_k = x_k$ and $i = k$.

First assume $a_k = x_k$ and $i = k$. Assume $y \sim x_k, y_j \sim a_j$ are such that $d_{\bar{a};k,j}^*(y, a_j) = m + 1 = d_{\bar{a};k,j}^*(x_k, y_j)$. Then (a_1, \dots, a_n) is geodesic with (x_1, \dots, x_n) . Lemma 2.4 implies that $y_j \sim a_j$ lies between a_j and x_j and hence $(a_1, \dots, a_{j-1}, y_j, a_{j+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) . Hence $d_{\bar{a};k,j}^*(y, y_j) = m + 2$.

Next we assume $a_k = y$ and $i = k$. Assume $y_k = x_k \sim y, y_j \sim a_j$ are such that $d_{\bar{a};k,j}^*(x_k, a_j) = m + 1 = d_{\bar{a};k,j}^*(y, y_j)$. If $d_{\bar{a};k,j}^*(x_k, y_j) < d_{\bar{a};k,j}^*(y, y_j)$, then $(a_1, \dots, a_{k-1}, x_k, a_{k+1}, \dots, y_j, a_{j+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) . Now $d_{\bar{a};k,j}^*(x_k, a_j) = m + 1$ implies that also $(a_1, \dots, a_{k-1}, x_k, a_{k+1}, \dots, a_j, a_{j+1}, \dots, a_n)$ is geodesic with (x_1, \dots, x_n) , and hence $d_{\bar{a};k,j}^*(x_k, a_j) < d_{\bar{a};k,j}^*(y, a_j)$, a contradiction.

Now assume $a_k = y$ and $i, j \neq k$. Let $y_i \sim a_i, y_j \sim a_j$ be such that $d_{\bar{a};i,j}^*(y_i, a_j) = m + 1 = d_{\bar{a};i,j}^*(a_i, y_j)$. If $d_{\bar{a};i,j}^*(y_i, y_j) < d_{\bar{a};i,j}^*(a_i, y_j)$, then $(a_1, \dots, y_j, a_{j+1}, \dots, a_n)$ and hence (a_1, \dots, a_n) are geodesic with (x_1, \dots, x_n) (while $(a_1, \dots, y_i, a_{i+1}, \dots, a_n)$ is not geodesic with (x_1, \dots, x_n) .) Thus for the unique $b_i \sim a_i$ between a_i and x_i we have $d_{\bar{a};i,j}^*(b_i, y_j) = m + 2$. But since $A \cup \{y\}$ satisfies Condition 2. we must have $y_i = b_i$, a contradiction. \square

Lemma 2.7. *Suppose that $(A, d^*), (A \cup \{y\}, d^*) \in \mathcal{K}_n$. Then $A \cup \{y\}$ is an elementary good extension of A .*

Proof. Let $y \sim x_k$ for some (unique) $x_k \in A_k$. If $d_{\bar{x};k}^*(y) < d_{\bar{x};k}^*(x_k)$ for all $(x_1, \dots, x_n) \in A$, then $A \cup \{y\}$ is an extension of type 1. So suppose for some $(x_1, \dots, x_n) \in A$ we have $d_{\bar{x};k}^*(y) > d_{\bar{x};k}^*(x_k)$, which we may assume to be locally maximal. Let $(z_1, \dots, z_n) \in A$ with $z_k = x_k$ and $d_{\bar{z};k}^*(y) > d_{\bar{z};k}^*(x_k)$. We have to show that \bar{z} and \bar{x} are geodesic. Otherwise for some j on the path $\gamma = (x_j = y_0, \dots, y_m = z_j)$ there is some y_s such that

$$d_{\bar{x};j}^*(y_s) < d_{\bar{x};j}^*(y_{s-1}, d_{\bar{x};j}^*(y_{s+1}$$

contradicting Axiom 3. □

Corollary 2.8. *Suppose that $(A, d^*), (B, d^*) \in \mathcal{K}_n$ with $A \subseteq B$. Then A is good in B . □*

Lemma 2.9. *The class \mathcal{K}_n has the amalgamation property.*

Proof. Clearly it suffices inductively to prove this for $A \subseteq B, C$ where B, C are elementary good extensions of A , so $B = A \cup \{b\}, C = A \cup \{c\}, b \in B_i, c \in C_j$ with unique neighbours b', c' .

If $i = j$ and at least one of the extensions is of type 1., then $A \cup \{b, c\}$ with the codistance induced by B, C is in \mathcal{K}_n . So suppose both extensions are of type 2. In this case $A \cup \{b, c\}$ with the codistance induced by B, C is in \mathcal{K}_n unless $b' = c'$ and b' and c' increase the distance to the same locally maximal tuple. In this case we identify b and c and choose B as the amalgam.

Now suppose $i < j$. If B is an elementary good extension of A of the first kind, define d^* on $A \cup \{b, c\}$ by

$$d_{\bar{x};i,j}^*(b, c) = d_{\bar{x};i,j}^*(b', c) - 1.$$

Then $A \cup \{b, c\}$ is an elementary good extension of C of the first kind and is therefore in \mathcal{K}_n by Lemma 2.6.

Next suppose that B and C are elementary good extensions of A of the second kind and that b, c were attached to different locally maximal tuples. Let $(y_1, \dots, y_{i-1}, b', y_{i+1}, \dots, y_n)$ be the locally maximal tuples to which b was attached. If $(x_1, \dots, x_{i-1}, b', x_{i+1}, \dots, x_{j-1}, c, x_{j+1}, \dots, x_n)$ is geodesic with $(y_1, \dots, y_{i-1}, b', y_{i+1}, \dots, y_n)$ define d^* on $A \cup \{b, c\}$ by

$$d_{\bar{x};i,j}^*(b, c) = d_{\bar{x};i,j}^*(b', c) + 1.$$

and otherwise

$$d_{\bar{y};i,j}^*(b, c) = |d_{\bar{y};i,j}^*(b', c) - 1|.$$

Then $A \cup \{b, c\}$ is an elementary good extension of C of the second kind and is therefore in \mathcal{K}_n by Lemma 2.6.

Finally suppose that b, c are attached to the same locally maximal tuple $(x_1, \dots, x_{i-1}, b', x_{i+1}, \dots, x_{j-1}, c', x_{j+1}, \dots, x_n)$.

If $(y_1, \dots, y_{i-1}, b', y_{i+1}, \dots, y_{j-1}, c', y_{j+1}, \dots, y_n)$ is geodesic with

$$(x_1, \dots, x_{i-1}, b', x_{i+1}, \dots, x_{j-1}, c', x_{j+1}, \dots, x_n),$$

then

$$d_{\bar{y};i,j}^*(b, c) = d_{\bar{x};i,j}^*(b, c') + 1.$$

and otherwise

$$d_{\bar{y};i,j}^*(b, c) = |d_{\bar{y};i,j}^*(b, c') - 1|.$$

Then $A \cup \{b, c\}$ is an elementary good extension of C of the second kind and is therefore in \mathcal{K}_n by Lemma 2.6. □

Using a bit of model theory, we may extend the language of graphs with n -ary predicates $d_k^*, k \in \mathbb{N}$ denoting the codistance and by binary function symbols $f_i(x, y), i \in \mathbb{N}$ with $f_i(x, y) = z$ if z is the i^{th} element on the path from x to y if such a path of length at least i exists and $z = x$ otherwise. In this expanded language, the substructure generated by a subset A will include all finite geodesic paths between elements from the same coordinate tree A_k . Hence in this language, the class \mathcal{K}_n is closed under finitely generated substructures. Since by Axiom 1. any $A \in \mathcal{K}_n$ contains the structure consisting only of an n -tuple (x_1, \dots, x_n) with $d^*(\bar{x}) = 0$, the class \mathcal{K}_n has a Fraïssé limit M_n (see e.g. [6], Ch. 4.4), i.e. a countable structure whose finite substructures are exactly the structures isomorphic to elements of \mathcal{K}_n and whose automorphism group acts transitively on isomorphism classes of finite substructures.

Suppose A is an n -fold tree and $(x_1, \dots, x_n) \in A$ is such that $d^*(\bar{x}) = 0$. and $y, z \sim x_i$ for some $1 \leq i \leq n$. Then the set

$$\mathbb{T}_{\bar{x},y} = \bigcup_{1 \leq j \leq n} \{z \in A_j : d_{\bar{x};i,j}^*(y, z) > d_{\bar{x};i,j}^*(x_i, z)\}$$

is called a *half apartment*, and the set

$$\mathbb{T}_{\bar{x};y,z} = \mathbb{T}_{\bar{x};y} \cup \mathbb{T}_{\bar{x};z}$$

is called an *apartment*.

The automorphism group of a multiple tree $A = \{A_i\}_{i \leq n}$ is said to act *strongly transitively* if it acts transitively on the set of *marked apartments*, i.e. transitively on the set $\{(x_1, \dots, x_n, y) : x_i \in A_i, y \sim x_1\}$. This is equivalent to the automorphism group having a BN-pair (see [1] Ch. 6).

Hence we obtain our main theorem:

Theorem 2.1. *The Fraïssé limit M_n is a multiple tree whose automorphism group acts strongly transitively on the set of apartments.*

Proof. It is clear from the construction that if $d_{\bar{x};i}^*(x) = k$ then there exists a unique $y \sim x_i$ such that $d_{\bar{x};i}^*(y) = k + 1$.

The strong transitivity of $\text{Aut}(M_n)$ is immediate by the properties of the Fraïssé limit: the structure $\{(x_1, \dots, x_n)\} \cup \{y\}$ with $y \sim x_1$ is in \mathcal{K}_n and $\text{Aut}(M_n)$ acts transitively on the set of substructures of M_n isomorphic to it. \square

For a half apartment $\mathbb{T}_{\bar{x};y}$ the corresponding *root group* $U_{\mathbb{T},\bar{x};y}$ in $\text{Aut}(A)$ is the subgroup of $\text{Aut}(A)$ fixing the set $\mathbb{T}_{\bar{x};y} \cup \{v : v \sim z \text{ for some } z \in \mathbb{T}_{\bar{x};y}\}$ pointwise.

We say that a multiple tree satisfies the *Moufang condition* if for each root the corresponding root group acts transitively on the set of apartments containing the given root.

Remark 2.10. *It is easy to see that if a root group acts transitively on the set of apartments containing the given half-apartment, then it acts regularly. In other words, the stabilizer of an apartment inside a root group is trivial (see [4], Sec. 4).*

Proposition 2.11. *The automorphism group of the multiple tree M_n does not satisfy the Moufang condition.*

Proof. We claim that in fact all root groups are trivial. Note that by the properties of the Fraïssé limit, if (x_1, \dots, x_n) in M_n is such that $d^*(x_1, \dots, x_n) = 0$, the stabilizer H of x_1, \dots, x_n acts highly transitively (i.e. m -transitively for any m) on the set of vertices $z \sim x_i$ since for any $m \in \mathbb{N}$ the structure $\{x_1, \dots, x_n, z_1, \dots, z_m\}$ with $z_j \sim x_i, j = 1, \dots, m$ is in \mathcal{K}_n and d^* is uniquely determined. In particular, the stabilizer H_y of y in H acts highly transitively on the set of $z \sim x_i, z \neq y$ and normalizes $U_{\mathbb{T},\bar{x};y}$. If $U_{\mathbb{T},\bar{x};y}$ was nontrivial, this would contradict Remark 2.10, whence the claim. \square

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